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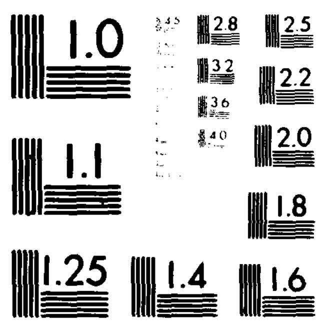

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THE DIAMETER OF ALMOST ALL BIPARTITE GRAPHS

by

V.L. Klee, D.G. Larman ~~and~~ E.M. Wright

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13. ABSTRACT

The diameter of a graph is of intrinsic interest as one of the most basic and most thoroughly studied parameters of graph theory. It may also be of practical concern because of the close relationship of diameters to the computational complexity of graph-theoretic algorithms based on breadth-first search. Here we consider bipartite graphs on n labelled points in one part and $m = m(n) \leq n$ labelled points in the other. Of the 2^{mn} such graphs, some are disconnected and the others have diameters ranging from $2m$ down to 2. Denoting by $P(n)$ the proportion of the graphs which are of diameter 3, we prove

Theorem 1. A necessary and sufficient condition that $P(n) \to 1$ as $n \to \infty$ is that

$$m(n) \log (4/3) - 2 \log n \to +\infty \ .$$

Some related results and unsolved problems are also mentioned.

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THE DIAMETER OF ALMOST ALL BIPARTITE GRAPHS

V.L. Klee, D.G. Larman and E.M. Wright*

Introduction

The diameter of a graph is of intrinsic interest as one of the most basic and most thoroughly studied parameters of graph theory. It may also be of practical concern because of the close relationship of diameters to the computational complexity of graph-theoretic algorithms based on breadth-first search [1]. Here we consider bipartite graphs on n labelled points in one part and $m = m(n) \leq n$ labelled points in the other. Of the 2^{mn} such graphs, some are disconnected and the others have diameters ranging from $2m$ or $2m-1$ down to 2. Denoting by $P(n)$ the proportion of the graphs which are of diameter 3, we prove

Theorem 1. <u>A necessary and sufficient condition that</u> $P(n) \to 1$ <u>as</u> $n \to \infty$ <u>is that</u>

$$m(n) \log (4/3) - 2 \log n \to +\infty . \tag{1}$$

Some related results and unsolved problems are also mentioned.

Proof of Theorem 1

In this section only, the term <u>graph</u> will mean a bipartite graph on labelled red points $a_1, \ldots, a_m$ in one part and labelled blue points $b_1, \ldots, b_n$ in the other. Each line joins a red point and a blue point; no pair of points is joined by more than one line. Thus there are 2^{mn} graphs. We use the phrases "proportion of these graphs which have a property" and "probability of a random graph having the property" as equivalent. Thus, for example, the probability of a particular line being present in a random graph is $\frac{1}{2}$.

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A graph has <u>diameter</u> k if every pair of its points is connected by a path of length not greater than k and at least one pair is not connected by a shorter path. We remark that only the complete graph has diameter 2, while the necessary and sufficient condition for any other graph to have diameter 3 is that every pair of copartite points in it should be joined by a 2-path. For, if this is not true for some copartite pair, the diameter is at least 4. Next, suppose that every copartite pair is joined by a 2-path and consider the pair of points a_i, b_j. If the line $a_i b_j$ is present, a 1-path suffices. (Only in the complete graph is this true for every pair a_i, b_j). If the line $a_i b_j$ is absent, some line $a_i b_k$ must be present, since a_i cannot be isolated. But b_k, b_j are joined by a 2-path and hence a_i, b_j are joined by a 3-path.

The probability that the blue points b_i, b_j are connected by a 2-path through a_k is $\frac{1}{4}$; the probability that they are not so joined is $\frac{3}{4}$. Hence the probability that none of the paths $b_i a_k b_j$ is present for any k is $(\frac{3}{4})^m$, i.e. the number of our graphs in which b_i, b_j (for a particular i, j) are not connected by a 2-path is $(\frac{3}{4})^m 2^{mn}$. There are $\frac{1}{2}n(n-1)$ pairs b_i, b_j. Hence, if we write L_1 for the "number" of graphs in which a pair b_i, b_j are not joined by a 2-path, each graph being counted as many times as the number of such pairs it contains, we have

$$L_1 = \frac{1}{2}n(n-1)(\tfrac{3}{4})^n 2^{mn} \tag{2}$$

and if we write $Q2^{mn}$ for the number of graphs which contain at least one such pair, then

$$Q2^{mn} \leq L_1 .$$

The number of graphs in which there is at least one red pair a_i, a_j not joined by a 2-path is by a similar argument not more than

$$\frac{1}{2}m(m-1)(\tfrac{3}{4})^n 2^{mn} \leq \frac{1}{2}n(n-1)(\tfrac{3}{4})^n 2^{mn} = o(2^{mn})$$

for all $m \leq n$. Hence the proportion of our graphs which have such a red pair is $o(1)$. These graphs can therefore be neglected and so Theorem 1 would follow at once from the following lemma.

Lemma. The necessary and sufficient condition that $Q \to 0$ as $n \to \infty$ is (1).

We now write L_2 for the "number" of our graphs which contain a couple of copartite blue pairs neither of which is connected by a 2-path. Each graph is counted according to the number of such couples that it contains. By the Inclusion-Exclusion Theorem,

$$L_1 - L_2 \le Q2^{mn} \le L_1. \tag{3}$$

It follows with the aid of (2) that

$$Q \le L_1 2^{-mn} = \tfrac{1}{2}n(n-1)(\tfrac{3}{4})^m \to 0$$

if (1) is satisfied, and this proves sufficiency in the lemma.

We have now to calculate L_2. Couples of different pairs of blue points are of two kinds. Typical of one kind is the couple which consists of the pair b_1, b_2 and the pair b_3, b_4; there are $\frac{1}{8}n(n-1)(n-2)(n-3)$ such couples. Typical of the other kind is the couple that consists of b_1, b_2 and b_1, b_3; there are $\frac{1}{2}n(n-1)(n-2)$ such couples.

The probability that the 2-path $b_1 a_k b_2$ is not present is $\frac{3}{4}$ and so is the independent probability that $b_3 a_k b_4$ is not present. Hence the probability that neither of these paths is present for any k is $(\frac{3}{4})^{2m}$. The position in the other case is a little more complicated; we must find the probability that neither $b_1 a_k b_2$ nor $b_1 a_k b_3$ is present, and the probabilities for the two paths are not independent. The probability that the line $b_1 a_k$ is absent is $\frac{1}{2}$. The probability that $b_1 a_k$ is present but both $a_k b_2$ and $a_k b_3$ are absent is $\frac{1}{8}$. Hence the probability that neither of the two paths $b_1 a_k b_2$ nor $b_1 a_k b_3$ is present is $\frac{5}{8}$. The probability that this is true for all k is $(\frac{5}{8})^m$, whence

$$L_2 = \{\tfrac{1}{8}(\tfrac{3}{4})^{2m} n(n-1)(n-2)(n-3) + \tfrac{1}{2}(\tfrac{5}{8})^m n(n-1)(n-2)\}2^{mn}.$$

If (1) is not true, there is an infinite sequence of values of n for which

$$m \log (4/3) - 2 \log n < A \tag{4}$$

for some A and so

$$L_1 > \frac{1}{4}e^{-A_2 mn} \quad . \tag{5}$$

For the present, we confine our attention to such a sequence of n. We take $A > \log 4$ and suppose that

$$\log 4 < m \log (4/3) - 2 \log n, \text{ i.e. } n^2 (\tfrac{3}{4})^m < \frac{1}{4} \; . \tag{6}$$

We have then

$$L_2/L_1 = \frac{1}{4}(\tfrac{3}{4})^m (n-2)(n-3) + (5/6)^m (n-2) \le \frac{1}{16} + \frac{1}{2} = \frac{9}{16} \; ,$$

since

$$(5/6)^{2m}(n-2)^2 \le n^2 (25/36)^m < n^2 (\tfrac{3}{4})^m < \frac{1}{4} \; .$$

Hence

$$L_1 - L_2 \ge 7L_1/16 > 7e^{-A_2 mn}/64$$

by (5) and so, by (3),

$$Q > 7e^{-A}/64 \; . \tag{7}$$

Now we fix n, remove the restriction (6) and consider $Q = Q(m)$. Note that $Q(m+1) \le Q(m)$, for the addition of the red point a_{m+1} to any (m,n) bipartite graph (with or without lines incident to a_{m+1}) cannot increase the probability that b_i, b_j are not joined by a 2-path through one of $a_1, \ldots, a_m$. Hence, since (7) holds when (6) holds, it holds also when the restriction (6) is withdrawn, i.e. for smaller m.

Hence (7) holds whenever (4) is satisfied. This proves necessity in the lemma and Theorem 1 follows.

Related results and problems

One of us has proved [7] that almost all labelled (m,n) bipartites have only the trivial automorphism provided that

$$m \log 2 - 4 \log n \to +\infty \; . \tag{8}$$

But

$$\log 2 > \log (16/9) = 2 \log (4/3)$$

and so (8) is certainly true if (1) is. Hence we have the following theorem.

Theorem 2. *If* (1) *is satisfied and* $m \leq n$, *almost all unlabelled* (m,n) *bipartite graphs have diameter* 3.

Now let us use the term $(m,n;E)$ *graph* to denote a labelled (m,n) bipartite having exactly E lines. The numbers m and E depend on n, with $E \leq mn$ and $m \to \infty$ as $n \to \infty$. In another paper [5] we obtain threshhold results (in the sense of Erdös and Rényi [2,3]) on the connectedness of $(m,n;E)$ graphs. These results may be regarded as dealing with diameters, since a graph is connected if and only if it is of finite diameter. However, there is a large gap between the case of finite diameter and the case of diameter 3 treated in Theorem 1. Much of the gap is filled by the following conjecture, which we have not proved.

Conjecture. *If* m, E, *and the positive integer* r *are such that*

$$E^{2r-2}/m^{r-1}n^{r} \to 0 \quad \text{and} \quad \left(E^{2r}/m^{r}n^{r+1}\right) - \log n \to \infty$$

as $n \to \infty$, *then the probability that a random* $(m,n;E)$ *graph is of diameter* $2r$ *or* $2r+1$ *converges to* 1 *as* $n \to \infty$.

Theorem 1 may be regarded as a bipartite analogue of Moon's and Moser's observation [6] that the probability that a random graph on labelled points $p_1, \ldots, p_n$ is of diameter 2 converges to 1 as $n \to \infty$. The conjecture is a bipartite analogue of the theorem, established in [4], that if d is an integer ≥ 2 such that $E^{d-1}/n^{d} \to 0$ and $(E^{d}/n^{d+1}) - \log n \to \infty$ as $n \to \infty$, then the probability that a random graph on $p_1, \ldots, p_n$ with E lines is of diameter d converges to 1 as $n \to \infty$. Probably the conjecture can be proved by adapting the computations in [4], but that would be a task of considerable technical difficulty.

For each (m,n) bipartite G on labelled points $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ let $G_a \langle$ resp. $G_b \rangle$ denote the graph whose points are $a_1, \ldots, a_m$ $\langle$resp. $b_1, \ldots, b_n \rangle$ and whose lines are those pairs xy of points which are joined by a 2-path in G. If $1 \leq m \leq n$ and $(\log n)/m \to 0$ as $n \to 0$, then the expected number of lines of G_a $\langle$resp. $G_b\rangle$ for a random $(m,n;E)$ graph is of the order of E^2/n $\langle$resp. $E^2/m\rangle$ as $n \to \infty$. If the appropriate independence (or asymptotic independence) results could be established, then G_a and G_b could be treated as random graphs on $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively and the conjecture would follow from the result of [4]. Even lacking this independence, the methods of [4] might be applicable.

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